

Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 16 May, 2023, 10:15

1. Let $A \in \mathbb{R}^{m \times n}$ have a singular value decomposition $A = U\Lambda V^T$.
 - (a) Show that the squares of the singular values of A are eigenvalues of the symmetric matrices $A^T A \in \mathbb{R}^{n \times n}$ and $AA^T \in \mathbb{R}^{m \times m}$. What are the corresponding eigenvectors? If $n > m$, what are the remaining $n - m$ eigenvalues of $A^T A$? If $m > n$, what are the remaining $m - n$ eigenvalues of AA^T ?
 - (b) Verify the identity $\text{Ran}(A)^\perp = \text{Ker}(A^T)$ using the singular value decomposition.
2. Let $A \in \mathbb{R}^{m \times n}$ and recall how the Moore–Penrose pseudoinverse $A^\dagger \in \mathbb{R}^{n \times m}$ is defined. Prove (some of) the identities

$$\begin{aligned}A^\dagger AA^\dagger &= A^\dagger, \\AA^\dagger A &= A, \\(A^\dagger A)^T &= A^\dagger A, \\(AA^\dagger)^T &= AA^\dagger.\end{aligned}$$

Using these, show that

$$\begin{aligned}A^\dagger A: \mathbb{R}^n &\rightarrow \text{Ker}(A)^\perp, \\AA^\dagger: \mathbb{R}^m &\rightarrow \text{Ran}(A)\end{aligned}$$

are orthogonal projections.

3. Consider the matrix equation $Ax = y$, where $A \in \mathbb{R}^{m \times n}$. The corresponding *least squares problem* is to find a *least squares solution* x_{LS} that minimizes the Euclidean norm of the residual, i.e.,

$$\|Ax_{\text{LS}} - y\| = \min_{x \in \mathbb{R}^n} \|Ax - y\| = \min_{z \in \text{Ran}(A)} \|z - y\|.$$

- (a) Show that $A^\dagger y$ is a least squares solution and it satisfies the *normal equation*

$$A^T Ax = A^T y.$$

Why is this solution special?

- (b) Show that $\text{Ker}(A^T A) = \text{Ker}(A)$.
- (c) Use the above results to deduce that $x \in \mathbb{R}^n$ is a least squares solution if and only if it satisfies the normal equation.