Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 3 June 2025, 10:15 am

1. Take a look at Subsection 1.5: “A new kind of example” in the *open access* article `https://doi.org/10.1017/S1446181112000077`

   The authors consider the function
   $$F(\boldsymbol{y}) := \frac{1}{1+\sum_{j=1}^s j^{-2}y_j^{\alpha}}, \quad \boldsymbol{y} := (y_1,\ldots,y_s)\in[0,1]^s, \quad \alpha\in(0,1].$$

   (a) Verify the authors’ claim that
   $$\frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}}F(\boldsymbol{y}) = |\mathfrak{u}|!F(\boldsymbol{y})^{|\mathfrak{u}|+1}\prod_{j\in\mathfrak{u}}(-\alpha j^{-2}y_j^{\alpha-1}) \quad \text{for all } \varnothing\neq\mathfrak{u}\subseteq\{1,\ldots,s\}.$$
   Note that here $\frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}} = \prod_{j\in\mathfrak{u}}\frac{\partial}{\partial y_j}$ for all $\varnothing\neq\mathfrak{u}\subseteq\{1,\ldots,s\}$ and $|\mathfrak{u}|$ denotes the cardinality of set $\mathfrak{u}\subseteq\{1,\ldots,s\}$.

   (b) Let $\boldsymbol{\gamma} := (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq\{1,\ldots,s\}}$ be a sequence of positive real numbers. During the lectures we considered an unanchored, weighted Sobolev space $H_{s,\boldsymbol{\gamma}}$ equipped with the norm
   $$\|f\|_{s,\boldsymbol{\gamma}}^2 = \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}}\frac{1}{\gamma_{\mathfrak{u}}}\int_{[0,1]^{|\mathfrak{u}|}}\bigg(\int_{[0,1]^{s-|\mathfrak{u}|}}\frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}}f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}_{-\mathfrak{u}}\bigg)^2\mathrm{d}\boldsymbol{y}_{\mathfrak{u}}, \quad f\in H_{s,\boldsymbol{\gamma}},$$
   where $\mathrm{d}\boldsymbol{y}_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}}\mathrm{d}y_j$ and $\mathrm{d}\boldsymbol{y}_{-\mathfrak{u}} := \prod_{j\in\{1,\ldots,s\}\setminus\mathfrak{u}}\mathrm{d}y_j$ for $\mathfrak{u}\subseteq\{1,\ldots,s\}$. Show that the function considered in part (a) satisfies
   $$\|F\|_{s,\boldsymbol{\gamma}}^2 \leq \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}}\frac{(|\mathfrak{u}|!)^2\prod_{j\in\mathfrak{u}}b_j^2}{\gamma_{\mathfrak{u}}}, \quad \text{where } b_j = \frac{\alpha}{j^2\sqrt{2\alpha-1}}.$$

2. Let $s\in\mathbb{N}$, let $f,g:\mathbb{R}^s\to\mathbb{R}$ be infinitely many times continuously differentiable, and let $\boldsymbol{\nu}\in\mathbb{N}_0^s$ be a multi-index. Show that
   $$\partial^{\boldsymbol{\nu}}(f(\boldsymbol{y})g(\boldsymbol{y})) = \sum_{\boldsymbol{m}\leq\boldsymbol{\nu}}\binom{\boldsymbol{\nu}}{\boldsymbol{m}}\partial^{\boldsymbol{m}}f(\boldsymbol{y})\partial^{\boldsymbol{\nu}-\boldsymbol{m}}g(\boldsymbol{y}) \quad \text{for all } \boldsymbol{y}\in\mathbb{R}^s.$$
   Here, we have used the following multi-index notations:
   $$\begin{aligned}\boldsymbol{\nu} &= (\nu_1,\ldots,\nu_s)\in\mathbb{N}_0^s\\ \boldsymbol{m} &= (m_1,\ldots,m_s)\in\mathbb{N}_0^s\\ \boldsymbol{m}\leq\boldsymbol{\nu} &\Leftrightarrow m_j\leq\nu_j \text{ for all } j\in\{1,\ldots,s\}\\ \partial^{\boldsymbol{\nu}} &= \prod_{j=1}^s\frac{\partial^{\nu_j}}{\partial y_j^{\nu_j}}\\ \binom{\boldsymbol{\nu}}{\boldsymbol{m}} &= \prod_{j=1}^s\binom{\nu_j}{m_j}.\end{aligned}$$

The exercises continue on the next page.

3. (a) Let $s \in \mathbb{N}$ and $\gamma_j \geq 0$ for all $j \in \{1, \ldots, s\}$. Show that

$$\sum_{\mathfrak{u} \subseteq \{1,\ldots,s\}} \prod_{j \in \mathfrak{u}} \gamma_j = \prod_{j=1}^{s} (1 + \gamma_j).$$

(b) Let $\mathfrak{u} \subseteq \{1, \ldots, s\}$. Show that

$$|\mathfrak{u}|! \leq \prod_{j \in \mathfrak{u}} j.$$

(c) Let $\ell \in \mathbb{N}_0$ and $c_j \geq 0$ for all $j \in \mathbb{N}$. Show that

$$\sum_{\substack{|\mathfrak{u}|=\ell \\ \mathfrak{u} \subseteq \mathbb{N}}} \prod_{j \in \mathfrak{u}} c_j \leq \frac{1}{\ell!} \bigg(\sum_{j=1}^{\infty} c_j \bigg)^{\ell}.$$

4. Let $\mathscr{F} := \left\{ \boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\nu}| := \sum_{j \geq 1} \nu_j < \infty \right\}$ denote the set of finitely supported multi-indices. Prove the *generalized Vandermonde identity*

$$\sum_{\substack{|\boldsymbol{m}|=\ell \\ \boldsymbol{m} \leq \boldsymbol{\nu}}} \binom{\boldsymbol{\nu}}{\boldsymbol{m}} = \binom{|\boldsymbol{\nu}|}{\ell} \quad \text{for all } \boldsymbol{\nu} \in \mathscr{F} \text{ and } 0 \leq \ell \leq |\boldsymbol{\nu}|.$$

Hint: You can use a simple combinatorial argument or induction with respect to the order of the multi-indices.