Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 20 May 2025, 10:15 am **(extended DL)**
Please make sure to return your source code for all programming tasks
There will be no exercise session on Tuesday 13 May 2025

1. Let $K\colon [0,1]\times[0,1]\to\mathbb{R}$ be defined by $K(x,y)=\min\{x,y\}$. Find the eigenvalues $\lambda_k\in\mathbb{R}$ and eigenfunctions $\psi_k\in L^2(0,1)$ of
$$\int_0^1 K(x,y)\psi_k(y)\,\mathrm{d}y=\lambda_k\psi_k(x) \tag{1}$$
for all $k=1,2,3,\ldots$
Hint: Differentiating the integral equation (1) twice on both sides with respect to x should yield an ODE of the form $\lambda\psi''(x)+\psi(x)=0$. This is a second order ODE with constant coefficients, which has a general solution of the form $\psi(x)=A\sin\big(\frac{x}{\sqrt{\lambda}}\big)+B\cos\big(\frac{x}{\sqrt{\lambda}}\big)$, $A,B\in\mathbb{R}$.

2. Let us consider the high-dimensional integral
$$I_s:=\int_0^1\cdots\int_0^1\cos\bigg(\sum_{i=1}^s x_i\bigg)\,\mathrm{d}x_1\cdots\mathrm{d}x_s.$$
Estimate the value of this integral by implementing a Monte Carlo sampler in your favorite programming language. That is, compute the sample average
$$Q_{s,n}(f)=\frac{1}{n}\sum_{k=1}^n f(\boldsymbol{t}_k),\quad f(\boldsymbol{x}):=f(x_1,\ldots,x_s):=\cos\bigg(\sum_{i=1}^s x_i\bigg),$$
where $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n$ is a random sample drawn from the uniform distribution $\mathcal{U}([0,1]^s)$.
In this case, the exact value of this integral is $I_s=2^s\cos\big(\frac{s}{2}\big)(\sin\frac12)^s$ (you do not need to prove this). Compute the error $|I_s-Q_{s,n}(f)|$ for $n=2^k$, $k=0,1,2,\ldots,20$. Try out several values for the dimension s, for example, $s=10,100,1\,000,\ldots$. What convergence rate do you observe for the error as a function of n? Does increasing the dimension s affect the convergence rate?

3. Let $D=(0,1)^2$, $f(\boldsymbol{x})=x_1$, and consider the following parametric PDE problem: for all $\boldsymbol{y}\in[-1/2,1/2]^s$, find $u(\cdot,\boldsymbol{y})\in H_0^1(D)$ such that
$$\int_D a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y})\cdot\nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_D f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\quad\text{for all } v\in H_0^1(D),$$
endowed with the (dimensionally-truncated) uniform and affine diffusion coefficient
$$a(\boldsymbol{x},\boldsymbol{y})=2+\sum_{k=1}^s y_k\psi_k(\boldsymbol{x}),\quad \boldsymbol{x}\in D,\ \boldsymbol{y}\in[-1/2,1/2]^s,$$
with stochastic fluctuations $\psi_k(\boldsymbol{x}):=k^{-2}\sin(\pi k x_1)\sin(\pi k x_2)$.

Consider the problem of approximating

$$\mathbb{E}[u(\boldsymbol{x},\cdot)] = \int_{[-1/2,1/2]^s} u(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

using the Monte Carlo method with stochastic dimension $s = 100$. That is, for several values of n, draw a random sample $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n$ from $\mathcal{U}([-1/2,1/2]^s)$ and compute

$$\mathbb{E}[u(\boldsymbol{x},\cdot)] \approx \frac{1}{n}\sum_{i=1}^{n} u(\boldsymbol{x},\boldsymbol{y}_i).$$

To solve the PDE numerically for each $\boldsymbol{y}_i$, you can modify the script `fem.py` available on the course webpage. Fix $s = 100$ and estimate the $L^2(D)$ error by using the Monte Carlo estimate corresponding to $n' \gg n$ as a reference solution. What convergence rate do you obtain?

4. Let $D = (0,1)^2$ and consider the following parametric *spectral eigenvalue* problem: for all $\boldsymbol{y} \in [-1/2,1/2]^s$, find the *smallest* eigenpair $(\lambda(\boldsymbol{y}), u(\cdot,\boldsymbol{y})) \in (\mathbb{R} \times (H_0^1(D) \setminus \{\boldsymbol{0}\}))$, $\|u(\cdot,\boldsymbol{y})\|_{L^2(D)} = 1$, such that

$$\int_D a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \lambda(\boldsymbol{y}) \int_D u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } v \in H_0^1(D),$$

endowed with the (dimensionally-truncated) uniform and affine diffusion coefficient

$$a(\boldsymbol{x},\boldsymbol{y}) = 2 + \sum_{k=1}^{s} y_k\psi_k(\boldsymbol{x}), \quad \boldsymbol{x} \in D,\ \boldsymbol{y} \in [-1/2,1/2]^s,$$

with stochastic fluctuations $\psi_k(\boldsymbol{x}) := k^{-2}\sin(\pi k x_1)\sin(\pi k x_2)$.

Consider the problem of approximating

$$\mathbb{E}[\lambda(\cdot)] = \int_{[-1/2,1/2]^s} \lambda(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \quad \text{and} \quad \mathbb{E}[u(\boldsymbol{x},\cdot)] = \int_{[-1/2,1/2]^s} u(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

using the Monte Carlo method with stochastic dimension $s = 100$. That is, for several values of n, draw a random sample $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n$ from $\mathcal{U}([-1/2,1/2]^s)$ and compute

$$\mathbb{E}[\lambda(\cdot)] \approx \frac{1}{n}\sum_{i=1}^{n}\lambda(\boldsymbol{y}_i) \quad \text{and} \quad \mathbb{E}[u(\boldsymbol{x},\cdot)] \approx \frac{1}{n}\sum_{i=1}^{n} u(\boldsymbol{x},\boldsymbol{y}_i).$$

To solve the PDE numerically for each $\boldsymbol{y}_i$, you can modify the script `fem.py` available on the course webpage. Fix $s = 100$ and estimate the Euclidean error of $\mathbb{E}[\lambda(\cdot)]$ and the $L^2(D)$ error of $\mathbb{E}[u(\boldsymbol{x},\cdot)]$ by using the Monte Carlo estimate corresponding to $n' \gg n$ as a reference solution. What convergence rate(s) do you obtain?

Remark: For tasks 3–4, you can modify the file `lognormal_demo.py` on the course page.