Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 29 April 2025, 10:15 am

1. (a) Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $\| \cdot \|$ is the induced norm. Prove that an inner product satisfies the so-called parallelogram identity
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in X.$$
   (b) Consider the space $C([a,b])$. Show that the sup-norm
$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$$
   is not determined by any inner product.
   (c) Similarly consider the $L^p$ norms
$$\|f\|_p = \left( \int_a^b |f(x)|^p \, \mathrm{d}x \right)^{1/p},$$
   where $1 \leq p < \infty$. Prove that if $p \neq 2$, then this norm is never determined by an inner product.

   *Hint:* Any norm induced by an inner product satisfies the parallelogram identity.

2. Let $H$ be a real Hilbert space. Recall that the *orthogonal complement* of any subset $M \subset H$ is defined as
$$M^\perp := \{ y \in H \mid \langle x, y \rangle = 0 \text{ for all } x \in M \}.$$
   (a) Show that for any subset $M \subset H$, $M^\perp$ is a closed subspace of $H$ and $M \subset (M^\perp)^\perp$.
   (b) If $M$ is a subspace of $H$, show that $(M^\perp)^\perp = \overline{M}$, where the bar denotes the closure of a set.

3. Let $H_1$ and $H_2$ be real Hilbert spaces and let $A \colon H_1 \to H_2$ be a continuous linear operator. Recall that the *kernel* of operator $A$ is defined as
$$\mathrm{Ker}(A) := \{ x \in H_1 \mid Ax = 0 \}$$
   and the *range* of operator $A$ is defined as
$$\mathrm{Ran}(A) := \{ y \in H_2 \mid y = Ax \text{ for some } x \in H_1 \}.$$
   (a) Show that $\mathrm{Ker}(A)$ is a *closed* subspace of $H_1$.
   (b) Show that $\mathrm{Ran}(A)$ is a subspace of $H_2$.

**The exercises continue on the next page.**

4. (a) Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space and $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is the induced norm. Prove the *polarization identity*

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \quad \text{for all } x, y \in X.$$

(b) Let $(H_1, \langle\cdot,\cdot\rangle_{H_1})$ and $(H_2, \langle\cdot,\cdot\rangle_{H_2})$ be real Hilbert spaces. Suppose that $U\colon H_1 \to H_2$ is a linear *isometry*: $\|Ux\|_{H_2} = \|x\|_{H_1}$ for all $x \in H_1$, where $\|\cdot\|_{H_1} := \sqrt{\langle\cdot,\cdot\rangle_{H_1}}$ and $\|\cdot\|_{H_2} := \sqrt{\langle\cdot,\cdot\rangle_{H_2}}$. Show that

$$\langle Ux, Uy\rangle_{H_2} = \langle x, y\rangle_{H_1} \quad \text{for all } x, y \in H_1.$$